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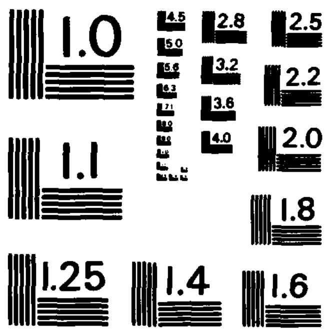

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

STOCHASTIC DIFFERENTIAL EQUATIONS
FOR NEURONAL BEHAVIOR

by

S.K. Christensen
and
G. Kallianpur

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In this article we extends the recent work of Kallianpur and Wolpert [5] modeling the behavior of neurons by means of stochastic partial differential equations on the dual of a nuclear space. The extensions will cover nuclear spaces of a more general structure and will apply to models described in terms of more general differential operators. A second objective of the present ~~our work~~ is to present a theoretical framework which will include the model recently proposed and heuristically investigated by Wan and Tuckwell in [9]. The authors ~~We~~ illustrate their ~~our~~ approach and its application by giving a rigorous treatment of the Wan & Tuckwell model. But first they ~~we shall~~ briefly describe the neurophysiological context. For a more detailed account, we refer to [5] and the references therein. In our description we shall follow the description

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STOCHASTIC DIFFERENTIAL EQUATIONS FOR NEURONAL BEHAVIOR

by

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Stochastic Differential Equations for Neuronal Behaviour

In this article we extend the recent work of Kallianpur and Wolpert [5] modeling the behavior of neurons by means of stochastic differential equations on the dual of a nuclear space. The extensions will cover nuclear spaces of a more general structure and will apply to models described in terms of more general differential operators. A second objective of our work is to present a theoretical framework which will include the model recently proposed and heuristically investigated by Wan & Tuckwell in [9]. We illustrate our approach and its application by giving a rigorous treatment of the Wan & Tuckwell model. But first we shall briefly describe the neurophysiological context. For a more detailed account, we refer to [5] and the references therein. In our description we shall follow the introduction in [5].

A neuron is a cell whose principal function is to transmit information along its considerable length, which often exceeds one meter. "Information" is represented by changing amplitudes of electrical voltage potentials across the cell wall. A quiescent neuron will exhibit a resting potential of about 60 mV, the inside more negative than the outside. Under certain circumstances the voltage potential in the neuron dendrite will rise above a threshold point at which positive feedback causes a pulse of up to 100 mV to appear at the base of the dendrite; this pulse is transmitted rapidly along the body and down the axon of the cell until it reaches the so-called "pre-synaptic terminals" at the other end of the neuron. Here the pulse causes tiny vesicles filled with chemicals called "neurotransmitters" to empty into the narrow gaps between the presynaptic terminals and the dentrites of other neurons. When these chemicals diffuse across the gap and hit the neighboring neurons' dendrites, they may cause the voltage potential in these dentrites to rise above a threshold point and initiate another pulse.

Let $\xi(t,x)$ represent the difference between the voltage potential at time t at the location $x \in X$ (= surface of the neuron) and the resting potential of about −60 mV. As time passes, ξ evolves due to two separate causes:

(i) Diffusion and leaks: Depending on the nature of X, the electrical properties of the cell wall may be approximated by postulating a contraction semigroup $\{T_t\}$ on $L^2(X,\Gamma)$ where Γ is a suitable σ-finite measure on X. For example, if $X = [0,b]$, core conductor theory suggest the semigroup corresponding to the diffusion equation

$$\frac{\partial \xi}{\partial t} = -\beta\xi_t + \delta\Delta\xi_t \quad (\beta,\delta > 0)$$

with Neumann (or insulating) boundary conditions at both ends. In neural material like heart muscle in which electrical signals can travel more easily in some directions than in others, the Laplacian should be replaced by a more general second-order elliptic operator.

(ii) Random fluctuations: Every now and then a burst of neurotransmitter will hit some place or another on the membrane and suddenly the membrane potential will jump up or down by a random amount at a random time and location. It is believed that these random jumps are quite small and quite frequent, making it reasonable to hope that they can be modeled by a Gaussian noise process; in any case the arrivals at distant locations or in disjoint time intervals are believed to be aproximately independent, justifying their modeling as a mixture of Poisson processes or as a generalised Poisson process.

Taking into account these random fluctuations one arrives at the following stochastic partial differential equation (SPDE) for :

$$d\xi(t,x) = A'\xi(t,x)dt + dM_t; \quad \xi_0 = \text{initial condition}$$

Here A is a suitable partial differential operator in spatial coordinates and M_t is an L^2-semimartingale. However, even for very simple choices of A (e.g. $A = I-\Delta$ in two dimensions; see [8]) a solution may exist only in the form of a generalized stochastic process i.e. a process taking values in the dual of a space Φ of "testfunctions". The relevant space of testfunctions can usually not be assumed to be the Schwartz space of all infinitely differentiable rapidly decreasing functions (see e.g. [5]) and therefore we shall take Φ to be a general countably Hilbert nuclear space. The linear SPDEs appropriate for this purpose have been investigated in [2] where an existence and uniqueness result is given for equations driven by a semimartingale on Φ'.

In [5] a restricted class of differential operators was considered; namely those which generate a selfadjoint contraction semigroup whose resolvent has a power which is Hilbert-Schmidt. In this case there is a canonical nuclear space upon which the SPDE has a very manageable form. However, the structure of the nuclear space is completely determined by the operator A, and therefore we shall present general results which are independent of the structure of Φ and which will also permit a wider class of differential operators to be considered.

Kallianpur and Wolpert ([5]) used a Poisson process N(AxBx(0,t]) to represent the number of voltage pulses of size $a \in A$ arriving at sites $x \in B \subset X$ (= surface of the neuron) at times prior to t.

Here, we adopt the point of view that, in practice, one can only "average" over the sites. Therefore it seems more realistic to assume that the arrival sites are given by "generalized functions" (distributions) $\eta \in \Lambda \subset \Phi'$, rather than by points x on the surface of the neuron membrane X. As we shall see, this approach will also offer the advantage of enlarging the class of possible models.

To pursue this idea let us consider a real rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$ (see Gel'fand & Vilenkin [3] p 79 for definition). Let $\mathcal{B}(\Phi')$ denote the Borel σ-field on Φ' and recall that $\mathcal{B}(\Phi')$ is the same whether we use the weakly or the strongly open sets in Φ' to define it.

-To avoid possible confusion with inner products we shall adopt the notation that for $\phi \in \Phi$ and $\eta \in \Phi'$, $\eta[\phi]$ will denote the value of the functional η evaluated at ϕ.

Let $\Lambda \in \mathcal{B}(\Phi')$ and let, for each $n \in \mathbb{N}$, μ^n be a σ-finite positive measure on $(\mathbb{R}\times\Lambda, \mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda))$ satisfying:

The mapping: $Q^n : \Phi\times\Phi \to \mathbb{R}$ defined by

$$Q^n(\phi,\psi) = \int_{\mathbb{R}\times\Lambda} a^2\eta[\phi]\eta[\psi]\mu^n(da\,d\eta)$$ is continuous on $\Phi\times\Phi$.

Let N^n be a Poisson random measure on $(\mathbb{R}\times\Lambda\times[0,\infty); \mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda)\times\mathcal{B}([0,\infty))$ with intensity measure $\mu^n(da\,d\eta)dt$ ($a \in \mathbb{R}$, $\eta \in \Lambda$, $t \in [0,\infty)$) (see e.g. Ikeda and Watanabe [4], page 42).

Let $\tilde{N}^n(da\,d\eta\,ds) = N^n(da\,d\eta\,ds) - \mu^n(da\,d\eta)ds$ and put

$$\tilde{Y}^n_t(\phi) = \int_{\mathbb{R}\times\Lambda\times[0,t]} a\eta[\phi]\tilde{N}^n(da\,d\eta\,ds); \quad \phi \in \Phi.$$

Let $m^n \in \Phi'$, and define $\tilde{X}^n_t(\phi) = tm^n[\phi] + \tilde{Y}^n_t(\phi)$; $\phi \in \Phi$.

Then, for each $\phi \in \Phi$, $\tilde{X}^n_t(\phi)$ is a real CADLAG semimartingale satisfying

$$E(\tilde{X}^n_t(\phi))^2 = t^2m^n[\phi]^2 + tQ^n(\phi,\phi).$$

Since Q^n is continuous on $\Phi\times\Phi$, the Kernel theorem for nuclear spaces (see Gel'fand & Vilenkin [3], page 74) yields the existence of $r(n) \in \mathbb{N}$ and $C(n) > 0$ such that

$$m^n[\phi]^2 + Q^n(\phi,\phi) \leq C(n)\|\phi\|^2_{r(n)}, \quad \forall\ \phi \in \Phi.$$

We shall henceforth assume that the same r and C will do for all $n \in \mathbb{N}$, i.e. we suppose that there exists $r_2 \in \mathbb{N}$, $C > 0$ such that

$$(1) \qquad m^n[\phi]^2 + Q^n(\phi,\phi) \leq C\|\phi\|^2_{r_2} \quad \forall\ n \in \mathbb{N} \quad \forall\ \phi \in \Phi.$$

From Doob's Martingale inequality we deduce that, for any $T > 0$,

$$E\sup_{0\leq t\leq T} (\tilde{X}^n_t(\phi))^2 \leq 2C(4T + 2T^2)\|\phi\|^2_{r_2} \quad \forall\ n \in \mathbb{N} \quad \forall\ \phi \in \Phi$$

and therefore Theorem III.1.12 and Remark 7 of [2] yields the existence of $q \in \mathbb{N}$, $q \geq r_2$ (independent of n) and a Φ_{-q}-valued CADLAG L^2-semimartingale X^n_t satisfying $X^n_t[\phi] = \tilde{X}^n_t(\phi)$ $\forall$ $t\geq 0$ (a.s.) $\forall$ $\phi \in \Phi$. Let $X^{n,T}_t := (X^n_t)_{t\in[0,T]}$; $T > 0$.

Let $m \in \Phi'$ and let $Q : \Phi\times\Phi \to \mathbb{R}$ be a continuous bilinear symmetric functional satisfying

$$(2) \qquad m[\phi]^2 + Q(\phi,\phi) \leq C\|\phi\|^2_{r_2}.$$

It can then be shown that there exists a Φ-valued process with independent increments and characteristic functional given by $\exp(itm[\phi] - t/2Q(\phi,\phi))$. This was shown by V. Perez-Abreu [7] for the case m=0 and a nuclear space of a special structure. The general case may be deduced from theorem III.1.12 in [2]. We shall henceforth call W a Φ'-valued Wiener process with parameters m and Q.
It can be shown from (1) and (2) (See [2] theorem III.1.12 and remark 7) that we may choose $q \geq r_2$ such that

$X^{n,T} \in D([0,T],\Phi_{-q})$ P-a.s. $\forall\, n \in \mathbb{N}$ $\forall\, T > 0$ and

$W^T \in C([0,T],\Phi_{-q})$ P-a.s. $\forall\, T > 0$.

Let P_T^n denote the measure induced on $D([0,T],\Phi_{-q})$ by $X^{n,T}$ and let P_T denote the measure induced on $C([0,T],\Phi_{-q}) \subset D([0,T],\Phi_{-q})$ by W^T.

I.0. PROPOSITION:

Let Q^n, m^n satisfy (1). Then, for every $T \geq 0$, the family $\{P_T^n : n \in \mathbb{N}\}$ is tight on $D([0,T],\Phi_{-q})$.

PROOF:

By Mitoma, [6], theorem 5.3.2. and remark R1, (page 996/7)it is sufficient to show that

(a) $\forall\, \phi \in \Phi : \{X^{n,T}[\phi] : n \in \mathbb{N}\}$ is tight on $D([0,T],\mathbb{R})$

and

(b) $\exists\, k \geq q : \forall\, \epsilon > 0\ \forall\, \rho > 0\ \exists\, \delta > 0 :$

$$\|\phi\|_k < \delta \Rightarrow P(\sup_{0 \leq t \leq T} |X_t^n[\phi]| > \epsilon) < \rho \quad \forall\, n \in \mathbb{N}$$

For part (a), by Billingsley [1] theorem 15.3 page 125, it is sufficient to show that $\forall\, \phi \in \Phi$:

(ai) $\forall\ \eta > 0\ \exists\ a > 0 : P(\sup_{0\le t\le T} |X_t^n[\phi]| > a) \le \eta \quad \forall\ n \in \mathbb{N}$

(aii) $\forall\ \epsilon > 0,\ \eta > 0\ \exists\ \delta \in (0,T)\ \exists\ n_o \in \mathbb{N}$:

$$P\{\sup_{t_1\le t\le t_2} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n - X_t^n[\phi]|\} \ge \epsilon\} \le \eta \quad \forall\ n \ge n_o$$

and

$$P(\sup_{s,t\in[0,\delta)} |X_s^n[\phi] - X_t^n[\phi]| \ge \epsilon) \le \eta \quad \forall\ n \ge n_o$$

and

$$P(\sup_{s,t\in[T-\delta,T)} |X_s^n[\phi] - X_t^n[\phi]| \ge \epsilon) \le \eta \quad \forall\ n \ge n_o$$

Fix $\phi \in \Phi$, and let $\eta > 0$, $\epsilon > 0$. Then,

$$P(\sup_{t\in[0,T]} |X_t^n[\phi]| > a) \le \frac{1}{a^2} E(\sup_{t\in[0,T]} |X_t^n[\phi]|)^2$$

$$\le \frac{2}{a^2} E(\sup_{t\in[0,T]} (t^2 m^n[\phi]^2 + \tilde{Y}_t^n[\phi]^2))$$

$$\le \frac{2C}{a^2}(T^2 m^n[\phi]^2 + 4TQ^n(\phi,\phi))$$

$$\le \frac{2}{a^2}(T^2 + 4T)\ C\|\phi\|_{r_2}^2 \quad \forall\ n \in \mathbb{N} \quad \text{(by 1)}$$

$$< \eta \text{ for } a^2 > \frac{2}{\eta}(T^2 + 4T)C\|\phi\|_{r_2}^2$$

Next, let $D:=\{t\ge 0:\ t_1 \le t \le t_2 \text{ and } t_2 - t_1 \le \delta\}$. Then

$$P\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n[\phi] - X_t^n[\phi]|\} \geq e\}$$

$$\leq e^{-2} E\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|, |X_{t_2}^n[\phi] - X_t^n[\phi]|\}\}^2$$

$$\leq e^{-2} E\{\sup_{t\in D} \min\{|X_t^n[\phi] - X_{t_1}^n[\phi]|^2, |X_{t_2}^n[\phi] - X_t^n[\phi]|^2)$$

$$\leq e^{-2} \min(E\sup_{t\in D} |X_{t-t_1}^n[\phi]|^2, E\sup_{t\in D} |X_{t_2-t}^n[\phi]|^2)$$

$$\leq \frac{2}{e^2}\min\{\sup_{t\in D}((t - t_1)^2 m^n[\phi]^2) + E\sup_{t\in D} |Y_{t-t_1}^n[\phi]|^2,$$

$$\sup_{t\in D}((t_2 - t)^2 m^n[\phi]^2) + E\sup_{t\in D} |Y_{t_2-t}^n[\phi]|^2\}$$

$$= \frac{2}{e^2}(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi))$$

$$\leq \frac{2}{e^2}(\delta^2 + 4\delta)C\|\phi\|_{r_2}^2 \quad \forall\, n \geq 1 \quad \text{(by (1))}$$

$$< \eta\ \forall\, n \in \mathbb{N} \text{ if } (\delta^2 + 4\delta) < (\frac{2}{e^2\eta}C\|\phi\|_{r_2}^2)^{-1}.$$

Further,

$$P(\sup_{s,t\in[0,\delta)} |X_s^n[\phi] - X_t^n[\phi]| \geq e)$$

$$\leq \frac{1}{e^2}E(\sup_{s,t\in[0,\delta)} |X_s^n[\phi] - X_t^n[\phi])^2$$

$$\leq \frac{1}{e^2}2(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi)) \leq \frac{2}{e^2}(\delta^2 + 4\delta)C\|\phi\|_{r_2}^2 \quad \forall\, n \geq 1$$

$$\leq \eta \ \forall\ n \in \mathbb{N} \text{ if } \delta^2 + 4\delta \leq (\frac{2}{\eta e^2} C \|\phi\|^2_{r_2})^{-1}.$$

Similarly,

$$P(\sup_{s,t \in [T-\delta,T]} |X^n_s[\phi] - X^n_t[\phi]| \geq e)$$

$$\leq 2e^{-2}(\delta^2 m^n[\phi]^2 + 4\delta Q^n(\phi,\phi)) \leq 2e^{-2}(\delta^2 + 4\delta) C \|\phi\|^2_{r_2} \quad \forall\ n \geq 1$$

$$\leq \eta \ \forall\ n \in \mathbb{N} \quad \text{if} \quad \delta^2 + 4\delta < (\frac{2}{e^2 \eta} C \|\phi\|^2_{r_2})^{-1}.$$

Hence (ai) and (aii) are satisfied for
$\delta^2 + 4\delta < (2\eta^{-1} e^{-2} C \|\phi\|^2_{r_2})^{-1}$, and $n_o = 1$. This proves (a).

(b): Fix $\phi \in \Phi$ and let $e, \eta > 0$.

Then $P(\sup_{t \in [0,T]} |X^n_t[\phi]| \geq e) \leq e^{-2} E(\sup_{t \in [0,T]} |X^n_t[\phi]|)^2$

$$\leq e^{-2} 2(T^2 m^n[\phi]^2 + 4TQ^n(\phi,\phi))$$

$$\leq e^{-2} 2(T^2 + 4T) C \|\phi\|^2_{r_2} \leq e^{-2} 2(T^2 + 4T) C \|\phi\|^2_q \text{ (by (1))}$$

$$\leq \eta \ \forall\ n \geq 1 \text{ if } \|\phi\|^2_q \leq \delta^2 = \frac{\eta e^2}{2(T^2+4T)C}.$$

This completes the proof of Proposition I.0.

<u>I.1. THEOREM:</u>

Suppose that, in addition to assumption (1),

(3) $$Q^n(\phi,\phi) \underset{n\to\infty}{\longrightarrow} Q(\phi,\phi) \quad \forall\ \phi \in \Phi.$$

(4) $$\lim_{n\to\infty} \int_{\mathbb{R}\times\Lambda} |a(\eta[\phi])|^3 \mu^n(da\,d\eta) = 0 \quad \forall\ \phi \in \Phi.$$

(5) $$m^n[\phi] \underset{n\to\infty}{\longrightarrow} m[\phi] \quad \forall\ \phi \in \Phi.$$

Then, for any $T > 0$, we have

$$P_T^n \underset{n\to\infty}{\Longrightarrow} P_T.$$

PROOF:

Fix $T > 0$. Let $t_1 \le t_2 \le \ldots \le t_K \in [0,T]$ and $\psi_1,\ldots,\psi_K \in \Phi$ for $K \in \mathbb{N}$ fixed.

We must show that

(i) $(X^n_{t_k}[\psi_k])_{k=1}^K$ converges in distribution to $(W_{t_k}[\psi_k])_{k=1}^K$

(ii) $\{P_T^n : n \in \mathbb{N}\}$ is tight on $D([0,T],\Phi_{-q})$.

(i): The log characteristic function of $(X^n_{t_k}[\psi_k])_{k=1}^K$ is:

$$C(a_1,\ldots a_K) = \Big[i \sum_{k=1}^{K} t_k a_k m^n[\psi_k] +$$

$$\int_0^T \int_{\mathbb{R}\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu^n(da\,d\eta)ds \Big]$$

where $F(s) := \sum_{k=1}^{K} a_k 1_{[0,t_k]}(s)\psi_k$,

while that of $(W_{t_k}[\psi_k])_{k=1}^{K}$ is

$$C(a_1,\ldots a_K) = \Big[i \sum_{k=1}^{K} t_k a_k m[\psi_k] - \frac{1}{2}\int_0^T Q(F(s),F(s))ds \Big].$$ Hence

$$|C_n(a_1,\ldots a_K) - C(a_1,\ldots,a_K)| = |i \sum_{k=1}^{K} t_k a_k (m^n[\psi_k] - m[\psi_k]) +$$

$$\int_0^T \Big[\int_{\mathbb{R}\times\Lambda} \sum_{p=3}^{\infty} (ia\eta[F(s)])^p \mu^n(da\,d\eta) - \frac{1}{2}(Q^n(F(s),F(s)) -$$

$$Q(F(s),F(s))) \Big] ds| \leq$$

$$|\sum_{k=1}^{K} t_k a_k (m^n[\psi_k] - m[\psi_k])| +$$

$$\int_0^T \int_{\mathbb{R}\times\Lambda} | \sum_{p=3}^{\infty} (ia\eta[F(s)])^p |\mu^n(da\,d\eta)ds +$$

$$\int_0^T \frac{1}{2}|Q^n(F(s),F(s)) - Q(F(s),F(s))|ds$$

$$\leq \sum_{k=1}^{K} t_k |a_k| |m^n[\phi_k]| + \int_0^T \int_{R\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta)ds +$$

$$\int_0^T \frac{1}{2} |Q^n(F(s),F(s)) - Q(F(s),F(s))|ds$$

the first term tends to zero by (5). As for the second term, use (4) to obtain

$$\text{(6)} \qquad \lim_{n\to\infty} \int_{R\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta) = 0 \quad \forall\ s \in [0,T].$$

Now, by definition of $F(s)$

$$|a\eta[F(s)]| \leq \sum_{k=1}^{K} |a||a_k||\eta[\phi_k]| \quad \forall\ s \in [0,T].$$

Define $a_k^* := a_k \operatorname{sign}(a_k \eta[\phi_k])$. Then

$$\sum_{k=1}^{K} |a||a_k||\eta[\phi_k]| = |a| \sum_{k=1}^{K} a_k^* \eta[\phi_k] = |a\eta[\sum_{k=1}^{K} a_k^* \phi_k]|$$

so

$$\text{(7)} \qquad |a\eta[F(s)]| \leq |a\eta[\sum_{k=1}^{K} a_k^* \phi_k]| \quad \forall\ s \in [0,T].$$

But $\sum_{k=1}^{K} a_k^* \phi_k \in \Phi$, so an application of (4) gives

$$\lim_{n\to\infty} \int_{R\times\Lambda} |a\eta[\sum_{k=1}^{K} a_k^* \phi_k]|^3 \mu^n(da\,d\eta) = 0 \quad \text{and thus}$$

$$\sup_{n\in N} \int_{R\times\Lambda} |a\eta[\sum_{k=1}^{K} a_k^* \phi_k]|^3 \mu^n(da\,d\eta) < \infty.$$

But then $\int_0^T \int_{R\times\Lambda} |a\eta[F(s)]|^3 \mu^n(da\,d\eta)ds \xrightarrow[n\to\infty]{} 0$

by (6), (7) and the DCT. Further,

$$Q^n(F(s),F(s)) \xrightarrow[n\to\infty]{} Q(F(s),F(s))$$

for each $s \in [0,T]$ by (3) and since Q and Q^n satisfy (1) we have

$$|Q^n(F(s),F(s)) - Q(F(s),F(s))| \le 2C\|F(s)\|_{r_2}^2.$$

Moreover, $\int_0^T \|F(s)\|_{r_2}^2\, ds < \infty$ so the DCT gives

$$\int_0^T |Q^n(F(s),F(s)) - Q(F(s),F(s))|ds \xrightarrow[n\to\infty]{} 0$$

concluding the proof of (i).

<u>(ii)</u> By (1) and Proposition 1.0 the family $\{P_T^n : n\in N\}$ is tight on $D([0,T],\Phi_{-q})$.

Let $A : \Phi \to \Phi$ be a linear and continuous, and suppose that A and $\{T_t : t \ge 0\}$ satisfies:

<u>A1</u> There exists a strongly continuous semigroup $[T_t: t\ge 0]$

on H whose generator coincides with A on Φ and such that:

(a) $T_t\Phi \subset \Phi \quad \forall\ t>0$

(b) $T_t|_{\Phi} : \Phi \to \Phi$ is continuous in $(\Phi, \ )\quad \forall\ t > 0$

(c) $t \to T_t\phi$ is $\mathcal{T}$-continuous for every $\phi \in \Phi$.

It is shown in [2] (Theorem III.1.5) that the SDE on Φ'

$$dX_t = A'X_t dt + dM_t; \quad X_o = Y,$$

where Y is a Φ'-valued random variable and $M = (M_t)_{t\geq 0}$ is a Φ'-valued weak L^2-semimartingale, i.e. a Φ'-valued process such that $M_t[\phi]$ is a semimartingale with $E(M_t[\phi])^2 < \infty\ \forall t\geq 0$ for each $\phi \in \Phi$, has a unique CADLAG solution on Φ'.

Moreover, the following result, which we restate for the convenience of the reader, is proved ([2], Theorem III.2.1):

I.2.THEOREM

Let M^n and M be weak Φ'-valued semimartingales satisfying

(*) $\forall\ T>0\ \exists\ q_T \in \mathbb{N}_o\ \forall\ n \in \mathbb{N} :\ M^{n,T},\ M^T \in D([0,T], \Phi_{-q_T})$

and

$$\sup_{n\in\mathbb{N}} E \sup_{0\leq t\leq T} \| M_t^{n,T} \|_{-q_T}^2 < \infty.$$

Let Y^n, Y be Φ'-valued random variables satisfying

(**) $\exists\, r_1 \in \mathbb{N}_o : \sup_{n\in\mathbb{N}} \max\{E\|Y^n\|^2_{-r_1}, E\|Y\|^2_{-r_1}\} < \infty.$

Let $T>0$ and suppose that $M^{n,T} \Rightarrow M^T$ on $D([0,T],\Phi_{-q_T})$ as $n\to\infty$

and that

$Y^n \Rightarrow Y$ as $n\to\infty$. Then there exists $p_T \in \mathbb{N}_o$ such that $\xi^{n,T} \Rightarrow \xi^T$ on $D([0,T],\Phi_{-p_T})$ as $n\to\infty$, where

$\xi^{n,T}$ and ξ^T denote the unique CADLAG solutions to

$$d\xi^n_t = A'\xi^n_t dt + dM^n_t;\quad \xi^n_o = Y^n,\quad 0\le t\le T$$

respectively

$$d\xi_t = A'\xi_t dt + dM_t;\quad \xi_o = Y,\quad 0\le t\le T.$$

Moreover, Theorem I.2 remains true if the spaces $D([0,T], \Phi_{-q_T}]$ and $D([0,T], \Phi_{-q_T}]$ are replaced by, $C([0,T], \Phi_{-q_T}]$, respectively $C([0,T], \Phi_{-q_T}]$; see [2].

Let ξ_n and η^o be Φ'-valued random variables, and let $\xi^n = (\xi^n_t)_{t\ge 0}$ denote the unique solution to the SDE on Φ'

$$d\xi^n_t = A'\xi^n_t dt + dX^n_t;\ \xi^n_o = \xi_n$$

and let $\eta = (\eta_t)_{t\ge 0}$ denote the unique solution to

$$d\eta_t = A'\eta_t dt + dW_t; \quad \eta_o = \eta^o$$

Remark 1

If the operator A is selfadjoint and dissipative (regarded as a densely defined linear operator on H) with $(I-A)^{-r_1}$ being a Hilbert-Schmidt operator for some $r_1 > 0$, and if Φ is the nuclear space generator by (I-A) (i.e. $\Phi = \{\phi \in H: \|(I-A)^r\phi\|_H^2 < \infty \ \forall \ r \in \mathbb{R}$; see [5]) then the solution may be expanded as a series

$$\eta_t = \sum_{j=1}^{\infty} \eta_t^j \phi_j$$

(Converging uniformly on [0,T] in the Φ_{-q} - topology (a.s.) for every T > 0 and $q \geq r_1 + r_2$; where $(\phi_j, -\lambda_j)$; $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$, is the eigensystem for A, and where η_t^j is the one-dimensional Ornstein-Uhlenbeck process given by

$$d\eta_t^j = -\lambda_j \ \eta_t^j dt + dW_t[\phi_j]; \ \eta_0^j = \eta^0[\phi_j] \ .$$

A similar expansion is possible for X_t^n in this case. We refer to [5] for details.

Remark 2

Regardless of the structure of Φ one can show that whenever A satisfies A1, for every T > 0 there exists $p_T \geq 0$ such that

$$\eta^T \in C([0,T], \Phi_{-p_T}) \ \text{(a.s.)}$$

where $C([0,1], \Phi\text{-}p_T)$ denotes the complete metric space of all continuous functions $f: [0,T] \to \Phi_{-p_T}$ and where $\eta^T := (\eta_t)_{t\in[0,T]}$.

1.3 THEOREM

Suppose that, in addition to (1),

(8) $\quad Q^n(\phi,\phi) \xrightarrow[n\to\infty]{} Q(\phi,\phi) \quad \forall\, \phi \in \Phi$

(9) $\quad \lim\limits_{n\to\infty} \int_{\mathbb{R}\times\Lambda} |a(\eta[\phi]|^3 \mu^n(da\,d\eta) = 0 \quad \forall\, \phi \in \Phi$

(10) $\quad \exists\, r \in \mathbb{N}: \sup\limits_n \max\{E\|\eta^o\|^2_{-r}, E\|\xi_n\|^2_{-2}\} < \infty$ and $\xi_n \Rightarrow \eta^o$ on Φ_{-r} as $n \to \infty$.

(11) $\quad m^n[\phi] \xrightarrow[n\to\infty]{} m[\phi] \quad \forall\, \phi \in \Phi$.

Then, for any $T > 0\ \exists\, p_T \in \mathbb{N}$:

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^T \text{ on } D([0,T], \Phi_{-p_T})$$

where $\xi^{n,T} = (\xi^n_t)_{t\in[0,T]}$ and

$\eta^T = (\eta_t)_{t\in[0,T]}$.

PROOF:

(1), (8), (9) and (10) imply that $X^{n,T} \underset{n\to\infty}{\Longrightarrow} W^T$ on $D([0,T],\Phi_{-q})$ $\forall$ $T \geq 0$ by theorem I.1. Moreover, (1) implies condition (*) Theorem I.2 while (10) supplies the remaining assumption of Theorem I.2, from which the conclusion is therefore obtained.

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Next, we shall give conditions under which the processes $X^{n,T}$ will converge weakly on $D([0,T],\Phi_{-q})$ to a process X^T constructed from a Poisson random measure N on $\mathbb{R}\times\Lambda\times[0,\infty)$ in the same way as X^n was constructed from N^n. We shall then invoke theorem I.2 to give sufficient conditions for the weak convergence of $\xi^{n,T}$ on $D([0,T],\Phi_{-p_T})$ to the solution to the SDE driven by X.

Let $m \in \Phi'$ and let μ be a σ-finite measure on $(\mathbb{R}\times\Lambda, \mathcal{B}(\mathbb{R}\times\mathcal{B}(\Lambda))$ satisfying

(11a) $\quad m[\phi]^2 + B(\phi,\phi) \leq C\|\phi\|_{r_2}^2 \qquad \forall\ \phi \in \Phi$

(11b) $\quad \int_{\mathbb{R}\times\Lambda} |e^{ia\eta[\phi]} - 1 - ia\eta[\phi]|\mu(dad\eta) < \infty$

where

$$B(\phi,\phi) := \int_{\mathbb{R}\times\Lambda} a^2\eta[\phi]^2\mu(dad\eta); \qquad \forall\ \phi \in \Phi.$$

Let N be a Poisson random measure on $(\mathbb{R}\times\Lambda\times[0,\infty), \mathcal{B}(\mathbb{R}\times\mathcal{B}(\Lambda)\times\mathcal{B}([0,\infty)))$ with intensity measure $\mu(dad\eta)dt$ ($a \in \mathbb{R}$, $\eta \in \Lambda$, $t \geq 0$).

Define

$$\tilde{Y}_t(\phi) = \int_{\mathbb{R}\times\Lambda\times[0,T]} a\eta[\phi](N(da\,d\eta\,ds) - \mu(da\,d\eta)ds);\ t \geq 0;\ \phi \in \Phi,$$

and $\tilde{X}_t(\phi) = tm[\phi] + \tilde{Y}_t(\phi)$.

Since the r_2 required in (11a) is the same as that of (1), theorem III.1.12 (b) of [2] implies the existence of a Φ_{-q} valued semimartingale $X = (X_t)_{t\geq 0}$ satisfying $X_t[\phi] = \tilde{X}_t(\phi)$ (a.s.) for every $\phi \in \Phi$. For each $T > 0$ let R_T denote the measure induced on $D([0,T),\Phi_{-q})$ by $X^T = (X_t)_{t\in[0,T]}$. Then we have:

I.4. THEOREM

Let m^n and μ^n satisfy (1). Let m,μ satisfy (11a,b) and suppose that

$$(12)\quad \int_{\mathbb{R}\times\Lambda} (e^{ia\eta[\phi]} - 1 - ia\eta[\phi])\mu^n(da\,d\eta) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}\times\Lambda}(e^{ia\eta[\phi]} - 1 - ia\eta[\phi])\mu(da\,d\eta) \quad \forall\ \phi \in \Phi$$

$$(13)\quad m^n[\phi] \longrightarrow m[\phi] \quad \forall\ \phi \in \Phi.$$

Then, for every $T > 0$, $P_T^n \Longrightarrow R_T$ as $n \to \infty$.

PROOF:

Fix $T > 0$. Since (1) is assumed to hold $\{P^n_T : n \geq 1\}$ is tight on $D([0,T],\Phi_{-q})$ by Proposition I.0. Hence it suffices to show finite dimensional convergence:

Let $0 \leq t_1 \leq \ldots \leq t_K \leq T$ and $\phi_k \in \Phi$; $k = 1,\ldots,K$.
Then the characteristic functions for $(X^{n,T}_{t_1}[\phi_1],\ldots,X^{n,T}_{t_K}[\phi_K])$ and $(X^T_{t_1}[\phi_1],\ldots,X^T_{t_K}[\phi_K])$ are, respectively,

$$C_n(a_1,\ldots,a_K) = \exp\Big[im^n[\sum_{k=1}^{K} t_k a_k \phi_k] +$$

$$\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu^n(da\,d\eta)ds\Big]$$

and

$$C(a_1,\ldots,a_k) = \exp\Big[im[\sum_{k=1}^{K} t_k a_k \phi_k] +$$

$$\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu(da\,d\eta)ds$$

where $F(s) := \sum_{k=1}^{K} a_k 1_{[0,t_k]}(s)\phi_k$. By (13) it is enough to show that

$$\lim_{n\to\infty} \exp\Big[\int_0^T \int_{R\times\Lambda}(e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu^n(da\,d\eta)ds\Big] =$$

$$\exp\Big[\int_0^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)]\mu(da\,d\eta)ds\Big] .$$

Now, $F(s)$ is piecewise constant, i.e. there are $0 = s_o <\ldots< s_M = T$ and $\phi_1,\ldots\phi_M \in \Phi$ such that

$$F(s) = \begin{cases} \phi_j \text{ if } s \in [s_{j-1}, s_j) & j = 1,\ldots,M-1 \\ \phi_M \text{ if } s \in [s_{M-1}, T] & \end{cases}$$

Hence

$$e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)] =$$

$$\sum_{j=1}^{M-1} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])1_{[s_{j-1},s_j)}(s) +$$
$$(e^{ia\eta[\phi_M]} - 1 - ia\eta[\phi_M])1_{[s_{M-1},T]}(s)$$

so

$$\int_o^T \int_{R\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)]\mu^n(dad\eta)ds =$$

$$\int_o^T \Big[\sum_{j=1}^{M-1} (\int_{R\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu^n(dad\eta)$$
$$1_{[s_{j-1},s_j)}(s) + \int_{R\times\Lambda} (e^{ia\eta[\phi_M]} - 1$$
$$- ia\eta[\phi_M])\mu^n(dad\eta)1_{[s_{M-1},T]}(s) \Big]ds =$$

$$\sum_{j=1}^{M} \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu^n(da\,d\eta)(s_j - s_{j-1})$$

(by (12)

$$\xrightarrow[n\to\infty]{} \sum_{j=1}^{M} \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[\phi_j]} - 1 - ia\eta[\phi_j])\mu(da\,d\eta)(s_j - s_{j-1})$$

(Recall that $\int_{\mathbf{R}\times\Lambda} \ldots\mu(da\,d\eta)$ is finite by (11b))

$$= \int_0^T \int_{\mathbf{R}\times\Lambda} (e^{ia\eta[F(s)]} - 1 - ia\eta[F(s)])\mu(da\,d\eta)ds,$$

concluding the proof. //

Let ξ^o be a Φ'-valued random variable and let $\xi = (\xi_t)_{t\geq 0}$ denote the unique solution to the Φ'-valued SDE

$$d\xi_t = A'\xi_t dt + dX_t; \quad \xi_o = \xi^o$$

I.5. THEOREM

Let m^n and μ^n satisfy (1), let m,μ satisfy (11a,b) and suppose that (12) and (13) hold. Suppose further that

(14) $\exists\ r \in \mathbf{N} : \sup_n \max\{E\|\xi_n\|^2_{-r}, E\|\xi^o\|^2_{-r}\} < \infty$

and that $\xi_n \underset{n\to\infty}{\Longrightarrow} \xi^o$ on Φ_{-r}.

Then, for any $T > 0$, $\exists\ p_T \in \mathbf{N}$:

$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \xi^T$ on $D([0,T],\Phi_{-p_T})$, where $\xi^T = (\xi_t)_{t\in[0,T]}$.

PROOF:

Let $T > 0$. Recall that $q \geq r_2$ is such that the canonical injection $i_q^{r_2}$ is Hilbert-Schmidt from $\Phi_q \to \Phi_{r_2}$. Let $\{\phi_j : j \in \mathbb{N}\}$ be a CONS in Φ_q consisting of elements of Φ. Then note that

$$E \sup_{0\leq t\leq T} \|X_t^{n,T}\|_{-q}^2 = E(\sup_{0\leq t\leq T} \sum_{j=1}^{\infty} (X_t^{n,T}[\phi_j])^2) \leq$$

$$E(\sum_{j=1}^{\infty} \sup_{0\leq t\leq T} (X_t^{n,T}[\phi_j])^2) = \sum_{j=1}^{\infty} E\sup_{0\leq t\leq T} (\tilde{X}_t^n[\phi_j])^2 \leq$$

$$\sum_{j=1}^{\infty} 2(T^2 m^n[\phi_j]^2 + 4TQ^n(\phi_j,\phi_j)) \leq \quad \text{(by (1))}$$

$$\sum_{j=1}^{\infty} 2C(T^2 \vee 4T)\|\phi_j\|_{r_2}^2 = 2C(T^2 \vee 4T)\|i_q^{r_2}\|_{HS}^2 \quad \forall\, n \in \mathbb{N},$$

(where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm) i.e. (*) of Theorem I.2 is satisfied. Moreover, $X^{n,T}, X^T \in D([0,T],\Phi_{-q})$ (P-a.s.) by assumption and X^n and X are Φ'-valued (weak) L^2-semimartingales. By Theorem I.4, (1), (11a,b), (12) and (13) imply that $X^{n,T} \underset{n\to\infty}{\Longrightarrow} X^T$ on $D([0,T],\Phi_{-q})$. Since also (14) is supposed to hold, the assumptions of Theorem I.2 are satisfied and the conclusion therefore follows from this theorem.

-Next we shall give conditions for the weak convergence of a sequence W^n of Φ'-valued Wiener processes to another Φ'-valued Wiener process W, and then employ these together with Theorem I.2 to giving the corresponding weak convergence result for the solutions to the SDE's driven by W^n and W, respectively.

Let, for $n \in \mathbb{N}$, $m^n \in \Phi'$ and let $B^n : \Phi x \Phi \to \mathbb{R}$ be bilinear symmetric functionals satisfying (1). Let $W^n = (W^n_t)_{t\geq 0}$ denote the Φ'-valued Wiener process with parameters m^n and B^n. (1) implies that $W^n_t \in \Phi_{-q}$ $\forall\ t \geq 0$, for some q which does not depend on $n \in \mathbb{N}$.

I.6. THEOREM

Suppose that, in addition to satisfying (1), B^n and m^n satisfy

$$(15) \qquad B^n(\phi,\phi) \xrightarrow[n\to\infty]{} Q(\phi,\phi) \quad \forall\ \phi \in \Phi$$

$$(16) \qquad m^n[\phi] \xrightarrow[n\to\infty]{} m[\phi] \quad \forall\ \phi \in \Phi.$$

Then, for each $T > 0$, we have $W^{n,T} \xRightarrow[n\to\infty]{} W^T$ on $C([0,T],\Phi_{-q})$,

where $W^{n,T} = (W^n_t)_{t\in[0,T]}$ and W_t is the Φ'-valued Wiener process introduced on page 6.

PROOF:

By Mitoma, [6] (Theorem 5.3 part 1), it is sufficient to prove that

$\forall\ t > 0\ \ \{W^{n,T} : n \in \mathbb{N}\}$ is tight on $C([0,T],\Phi_{-q})$ and

$\forall\ 0 \le t_1 \le \ldots \le t_N \le T\ \forall\ \phi_1,\ldots,\phi_N \in \Phi$:

$$(W^{n,T}_{t_j}[\phi_j])^N_{j=1} \underset{n\to\infty}{\Longrightarrow} (W^T_{t_j}[\phi_j])^N_{j=1}.$$

The tightness part is proved in the same way as in Proposition I.0.

Now, a calculation shows that

$$E\exp(i\sum_{j=1}^{N} a_j W^{n,T}_{t_j}[\phi_j]) =$$

$$\exp\left[i\sum_{j=1}^{N} t_j a_j m^n[\phi_j] - \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} t_j t_k a_j a_k B^n(\phi_j,\phi_k)\right]$$

$$\underset{n\to\infty}{\longrightarrow} \quad \text{(by (15) and (16))}$$

$$\exp\left[i\sum_{j=1}^{N} t_j a_j m[\phi_j] - \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} t_j t_k a_j a_k B(\phi_j,\phi_k)\right]$$

$$= E\exp(i\sum_{j=1}^{N} a_j W^T_{t_j}[\phi_j]).$$

$\Vert$

Letting $\eta^n = (\eta^n_t)_{t\ge 0}$ denote the unique solution to the SDE on Φ':

$$d\eta^n_t = A'\eta_t dt + dW^n_t \ ; \quad \eta^n_o = \eta_n$$

and $\eta = (\eta_t)_{t\geq 0}$ be the Φ'-valued process introduced on page 16 we have

I.7. THEOREM

Let, in addition to (1), B^n and m^n satisfy (15) and (16) of theorem I.6, and suppose that η_n and η^o satisfy

$$(17) \qquad \exists\ r \in \mathbb{N} : \sup_n \max\{E\|\eta_n\|_{-r}^2, E\|\eta^o\|_{-r}\} < \infty \text{ and}$$

$$\eta_n \underset{n\to\infty}{\Longrightarrow} \eta \text{ on } \Phi_{-r}.$$

Then, $\forall\ T > 0\ \exists\ p_T \in \mathbb{N}$: $\eta^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^T$ on $C([0,T],\Phi_{-p_T})$,

where $\eta^{n,T} := (\eta_t^n)_{t\in[0,T]}$.

PROOF:

By (1), (15) and (16) and theorem I.6 $W^{n,T} \underset{n\to\infty}{\Longrightarrow} W^T$ on $D([0,T],\Phi_{-q})$ $\forall\ t \geq 0$ where $q = \min\{p : i_p^{r_2}$ is Hilbert-Schmidt$\}$. Moreover (1) implies (*) of Theorem I.2 and (17) supplies the remaining condition of Theorem I.2 (recall remark 2).

As indicated at the beginning of this section, Kallianpur and Wolpert ([5]) used Poisson random measures defined via intensity measures on $(\mathbb{R}\times\mathcal{X}, \mathcal{B}(\mathbb{R})\times\mathcal{B})$ where $(\mathcal{X}, \mathcal{B})$ is a suitably chosen measurable space, rather than by mean/covariance measures defined on $(\mathbb{R}\times\Lambda, \mathcal{B}(\mathbb{R})\times\mathcal{B}(\Lambda))$;

$\Lambda \in \mathcal{B}(\Phi')$ as we have done it here.

It is therefore natural to address the question of when Kallianpur and Wolpert's framework is contained in the one we have presented here.

It is proved in [2] (Chapter IV) that if $H = L^2(X,T)$ for some - finite measure space (X,T) where X is a -compact topological Hausdorff space then it is possible to represent the Kallianpur & Wolpert models in our framework if elements of Φ are continuous functions on X and if, the evaluation functional $\delta_x : \phi \to \phi(x)$ is continuous on Φ for every $x \in X$.

II. THE WAN & TUCKWELL MODEL

Next, we shall apply our results to giving a rigorous formulation and investigation of a model recently proposed by Wan & Tuckwell [9]:

In order to study the behaviour of the difference $V(t,x)$ at time t between the so-called resting potential and the actual potential at point x on the surface of an infinitely thin cylinder shaped neuron which receives synaptic stimuli of the finite spatial extent ϵ_i at each of N sites x_i, Wan & Tuckwell investigated the model formally given by

$$(19) \quad \begin{cases} \dfrac{\partial V}{\partial t} = -V + \dfrac{\partial^2 V}{\partial x^2} + \sum\limits_{i=1}^{N} h(x;x_i,\epsilon_i)(\alpha_i + \beta_i \dfrac{dW^i}{dt}) \\ V(0,x) = 0 \quad V(t,0) = 0 = V(t,b); \quad \forall\, t \geq 0, \end{cases}$$

where

$$h(x;x_i,\epsilon_i) = 1_{(x_i-\epsilon_i,x_i+\epsilon_i)}(x)$$

($x_i,\epsilon_i > 0$ fixed for $i = 1,\ldots,N$)

and where W_t^i; $i = 1,\ldots,N$ are independent standard Wiener processes. α_i and β_i represent input current parameters and the neuron is thought of as the interval $[0,b]$; for some $b > 0$.

To see how this model can be given a rigorous representation as a Φ'-valued SDE, let $H = L^2([0,b])$ with inner product denoted by $\langle .,. \rangle_H$. Let L denote the operator $I - \Delta$ (Δ = Laplace operator in one dimension) with Neumann boundary conditions at 0 and b. Then L is a densely defined positive definite selfadjoint closed linear operator on H and admits a CONS $\{\phi_j : j = 0,1,2,\ldots\}$ in H consisting of eigenvectors of L;

$$L\phi_j = \lambda_j\phi_j;\ j = 0,1,2,\ldots,\ \text{where } \lambda_j = 1 + \frac{j^2\pi^2}{b^2} \text{ and}$$

$$\phi_j(x) = \begin{cases} b^{-1/2} & \text{if } j = 0 \\ \left(\frac{2}{b}\right)^{1/2} \mathrm{Cos}\left(\frac{j\pi x}{b}\right) & \text{if } j \geq 1. \end{cases}$$

Further, $A := -L$ is the generator of a selfadjoint contraction semigroup $\{T_t : t \geq 0\}$ on H whose resolvent $R(\lambda) = (\lambda I - A)^{-1}$ is Hilbert-Schmidt on H.

Letting

$$\Phi := \{\phi \in H : \|(I - A)^r \phi\|_H < \infty \quad \forall\, r \in \mathbb{R}\}$$

and defining norms $\|\cdot\|_r$; $r \in \mathbb{R}$ on Φ by

$$\|\phi\|_r := \|(I - A)^r \phi\|_H ; \ \phi \in \Phi$$

we put Φ_r equal to the $\|\cdot\|_r$-completion of Φ.

Then $\Phi = \bigcap_{r \in \mathbb{R}} \Phi_r$ and if τ denotes the Frechet topology on Φ generated by $\{\|\cdot\|_r : r \in \mathbb{R}\}$ (i.e. the projective limit topology on Φ)), then $(\Phi, \tau) \hookrightarrow H \hookrightarrow \Phi'$ (where Φ' denotes the strong dual of (Φ, τ)) is a rigged Hilbert space. Since $A = -L$, and L is a densely defined positive selfadjoint closed linear operator on H it is easily seen that A and $\{T_t : t \geq 0\}$ satisfy A1 of section I.

Moreover, $\{\phi_j : j \in \mathbb{N}\}$ Φ, Φ Dom(L) and per construction of Φ every element of Φ is an infinitely differentiable function. Let $N \in \mathbb{N}$ fixed, and for each $i = 1,\ldots,N$ let $\xi_i \in \Phi'$. Let ν_i; $i = 1,\ldots,N$ be σ-finite measures on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} a^2 \nu_i(da) < \infty \quad \forall\, i,$$

and let μ be the measure on $\mathbb{R} \times \Lambda$, where $\Lambda = \{\xi_i : i = 1,\ldots,N\}$, given by

$\mu = \sum_{i=1}^{N} \nu_i x \delta_{\xi_i}$; where δ_ξ is the point mass at ξ.

Define

$$Q(\phi,\psi) = \int_{\mathbb{R}\times\Lambda} a^2 \eta[\phi]\eta[\psi]\mu(da\,d\eta); \quad \phi,\psi \in \Phi$$

$$= \sum_{i=1}^{N} \int_{\mathbb{R}} a^2 \nu_i(da)\, \xi_i[\phi]\xi_i[\psi]$$

then Q is a continuous, bilinear symmetric functional on Φ, so for $m \in \Phi'$ given, let $W = W_t$ be the Φ'-valued (actually Φ_{-q} valued for some $q \in \mathbb{N}_o$) Wiener process with parameters m and Q.

Consider the SDE on Φ':

$$(20) \qquad d\eta_t = A'\eta_t dt + dW_t, \quad \eta_o = 0$$

Now, W is a weak Φ'-valued continuous L^2-semimartingale, and since A and $\{T_t : t \geq 0\}$ satisfy A1 there is a unique continuous Φ'-valued solution (from [2], Theorem III.1.5 and Remark 6) given by

$$\eta_t[\phi] = \int_o^t W_s[T_{t-s}A\phi]ds + W_t[\phi] \quad \forall\, \phi \in \Phi \text{ (with probability one).}$$

Choosing $\xi_i = \langle h(.;x_i,e_i),.\rangle_H \quad \forall\ i = 1,\ldots,N$ and

$$m = m^e := \sum_{i=1}^{N} \alpha_i \xi_i; \quad \beta_i^2 = \int_{\mathbb{R}} a^2 \nu_i(da),$$

(20) is the representation of (19) as an SDE on Φ'. To see that this is indeed the case, expand

$$\phi = \sum_{j=0}^{\infty} \langle \phi, \phi_j \rangle_H \, \phi_j \text{ (converging in } (\Phi, \mathcal{T}))$$

(recall that $\phi_j \in \Phi \quad \forall\, j \in \mathbb{N}$)

Then (writing η_t^ϵ for η_t and W_t^ϵ for W_t)

$$\eta_t^\epsilon[\phi] = \sum_{j=0}^{\infty} \left(\int_0^t W_s^\epsilon[T_{t-s} A\phi_j] ds + W_t^\epsilon[\phi_j] \right) \langle \phi, \phi_j \rangle_H$$

(converging in $L^2(\Omega, \mathcal{F}, P)$).

Moreover, the series

$$\sum_{j=0}^{\infty} \left[\int_0^t W_s^\epsilon[T_{t-s} A\phi_j] ds + W_t^\epsilon[\phi_j] \right] \phi_j(x)$$

is convergent in $L^2(\Omega, \mathcal{F}, P)$ for every $x \in [0,b]$ to a limit $V_\epsilon(t,x)$ satisfying

$$\eta_t^\epsilon[\phi] = \int_0^b V_\epsilon(t,x) \phi(x) dx \quad \text{(P-a.s.)}$$

for every $\phi \in \Phi$. Let us see that $EV_\epsilon(t,x)$ and $\mathrm{Var}V_\epsilon(t,x)$ actually agree with the formulae found in [9] by a heuristic argument:

$$EV_\epsilon(t,x) = E \sum_{j=0}^{\infty} \left(\int_0^t W_s[T_{t-s} A\phi_j] ds + W_t[\phi_j] \right) \phi_j(x)$$

$$= E \sum_{j=0}^{\infty} \left(\int_0^t -\lambda_j e^{-\lambda_j (t-s)} W_s[\phi_j] ds + W_t[\phi_j] \right) \phi_j(x)$$

$$= E\sum_{j=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)}dW_s[\phi_j]\phi_j(x)$$

$$= \sum_{j=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)}m[\phi_j]ds\phi_j(x)$$

$$= \sum_{j=0}^{\infty} m[\phi_j]\lambda_j^{-1}(1 - e^{-\lambda_j t})\phi_j(x)$$

$$= \sum_{i=1}^{N}\alpha_i \sum_{j=0}^{\infty}\frac{\phi_j(x)\Psi_j(x_i;\epsilon_i)}{\lambda_j}(1 - e^{-\lambda_j t})$$

which is formula (8) page 279 in Wan & Tuckwell [9]. Here, as in [9],

$$\Psi_j(x_i;\epsilon_i) = \langle h(.;x_i,\epsilon_i),\phi_j\rangle_H = \int_{x_i-\epsilon_i}^{x_i+\epsilon_i}\phi_j(x)dx.$$

Next,

$$\mathrm{Var}V_\epsilon(t,x) = E\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_0^t e^{-\lambda_j(t-s)}d\bar{W}_s[\phi_j]\cdot$$

$$\int_0^t e^{-\lambda_k(t-s)}d\bar{W}_s[\phi_k]\phi_j(x)\phi_k(x)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_0^t e^{-(\lambda_j+\lambda_k)(t-s)}Q(\phi_j,\phi_k)ds\phi_j(x)\phi_k(x)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{Q(\phi_j,\phi_k)}{\lambda_j+\lambda_k}\phi_j(x)\phi_k(x)(1 - e^{-(\lambda_j+\lambda_k)t})$$

$$= \sum_{i=1}^{N} \beta_i^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\phi_j(x)\phi_k(x)\Phi_j(x_i;\epsilon_i)\Phi_k(x_i;\epsilon_i)}{\lambda_j + \lambda_k} \cdot$$

$$(1 - e^{-(\lambda_j+\lambda_k)t})$$

which is formula (10) in [9].

Wan & Tuckwell proceed to compute the limit as $\epsilon_i \to 0 \ \forall \ i = 1,\ldots,N$ in such a way that $\epsilon_i \alpha_i \to a_i$ and $\epsilon_i \beta_i \to b_i > 0$ of $EV_\epsilon(t,x)$ and $VarV_\epsilon(t,x)$, and they find that these limits correspond to having point stimuli (i.e. $h(x,x_i,\epsilon_i)$ replaced by $\delta_{x_i}(x)$) at each of x_i; $i = 1,\ldots,N$.

This result may be obtained from Theorem I.7 in the following manner:

For each $i = 1,\ldots,N$ take $\nu_i^\epsilon = b_i \epsilon_i^{-1} \mu_i$, where μ_i a is finite measure on $\mathbb{R}$ with compact support.

Noting that every $\phi \in \Phi$ is a continuous function on $[0,b]$ (recall that $\Phi \subset \mathrm{Dom}(L)$ and that L is a differential operator) we let $\epsilon_i \to 0$ in such a way that $\epsilon_i \alpha_i \to a_i$. Then

$$\lim_{\epsilon_i \to 0} m_\epsilon[\phi] = \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \alpha_i \langle h(.;x_i,\epsilon_i),\phi\rangle_H$$

$$= \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \alpha_i \int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)\,dx$$

$$= \sum_{i=1}^{N} 2a_i \phi(x_i)$$

$$= \sum_{i=1}^{N} 2a_i \delta_{x_i}[\phi]$$

and

$$\lim_{\epsilon_i \to 0} Q^{\epsilon}(\phi,\phi) = \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} \beta_i^2 \left(\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)dx \right)^2$$

$$= \lim_{\epsilon_i \to 0} \sum_{i=1}^{N} b_i^2 \epsilon_i^{-2} \left(\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} \phi(x)dx \right)^2 \int_{\mathbb{R}} a^2 \ \mu_i(da)$$

$$= \sum_{i=1}^{N} 4b_i^2 \phi(x_i)^2 \int_{\mathbb{R}} a^2 \ \mu_i(da)$$

$$= \sum_{i=1}^{N} 4b_i^2 (\delta_{x_i}[\phi])^2 \int_{\mathbb{R}} a^2 \ \mu_i(da)$$

Also,

$$|m_{\epsilon}[\phi]|^2 + Q^{\epsilon}(\phi,\phi) \leq$$

$$\left[\left(\sum_{i=1}^{N} \| h(.;x_i,\epsilon_i) \|_H |\alpha_i| \right)^2 + \sum_{i=1}^{N} \beta_i^2 \| h(.,x_i,\epsilon_i) \|_H^2 \right] \|\phi\|_H^2$$

$$= \left(\sum_{i=1}^{N} 2\epsilon_i |\alpha_i| + \sum_{i=1}^{N} 4 \ b_i^2 \right) \|\phi\|_H^2$$

$\leq$ CONSTANT $\|\phi\|_H^2 \quad \forall\ \epsilon_i$,

since $\epsilon_i\alpha_i \to a_i$ and $\epsilon_i \to 0$; where CONSTANT is independent of ϵ_i, so condition (1) of section 1 is satisfied. Since the initial condition is zero, theorem I.7 yields

<u>Proposition</u>

As $\epsilon_i \to 0$ and $\epsilon_i\alpha_i \to a_i$ we have

$$\eta^{\epsilon_1 T} \Longrightarrow \eta^T \quad \text{on} \quad C([0,T],\ \Phi_{-q_T})\ \forall\ T > 0$$

for some $q_T > 0$, where $\eta = (\eta_t)_{t\geq 0}$ is the solution to (20) corresponding to

$$Q(\phi,\phi) = \sum_{i=1}^{N} 4\ b_i^2[\phi] \int_{\mathbb{R}} a^2\ \mu_i(da)$$

and

$$m[\phi] = \sum_{i=1}^{N} 2a_i\ \delta_{x_i}[\phi]\ .$$

Now, Take $\int_{\mathbb{R}} a^2\ \mu_i(da) = 1 \quad i=1,\dots,N.$ Then

$$E\eta_t[\phi] = \sum_{i=1}^{N} 2a_i \sum_{j=0}^{\infty} \frac{\langle\phi,\phi_j\rangle_H \phi_j(x_i)}{\lambda_j}(1 - e^{-\lambda_j t}); \qquad \phi \in \Phi$$

and

$$\mathrm{Var}\,\eta_t[\phi] = \sum_{i=1}^{N} 4b_i^2 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\langle\phi,\phi_j\rangle_H \langle\phi,\phi_k\rangle_H \phi_j(x_i)\phi_k(x_i)}{\lambda_j + \lambda_k}\ .$$

$$(1 - e^{-(\lambda_j+\lambda_k)t}).$$

Since $V_\varepsilon(t,x) = \sum_{j=0}^{\infty} \eta_t[\phi_j]\phi_j(x) \quad (\text{in } L^2(\Omega,\mathcal{F},P))$

we get

$$(22) \qquad EV_\varepsilon(t,x) = \sum_{i=1}^{N} 2a_i \sum_{j=0}^{\infty} \frac{\phi_j(x_i)}{\lambda_j}\phi_j(x)(1 - e^{-\lambda_j t})$$

and

$$(23) \qquad VarV_\varepsilon(t,x) = \sum_{i=1}^{N} 4b_i^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\phi_j(x_i)\phi_k(x_i)}{\lambda_j + \lambda_k}\phi_j(x)\phi_k(x).$$

(22) and (23) are the expressions found by Wan & Tuckwell for point stimuli at $x_i; i = 1,\dots,N$.

In practice, equation (20) is likely to arise as a limit of equations where the noise is not a Wiener process, but rather a process generated by a Poisson random measure in the manner considered in section I. As an illustration, take μ^n to be measures on $R\times\Lambda$; where

$\Lambda = \{\xi_i : i = 1,\dots,N\}$ of the form

$\mu^n = \sum_{i=1}^{N} \nu_i^n \times \delta_{\xi_i}$, where for each $n \in \mathbb{N}$ and $i = 1,\ldots,N$ ν_i^n is a σ-finite measure on $\mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} a^2 \nu_i^n(da) < C < \infty \quad \forall\, i = 1,\ldots,N.$$

Let $m^n \in \Phi'$ converge weakly to m_e. Then there is $r \in \mathbb{N}_o$ such that

$$|m^n[\phi]|^2 \leq K\|\phi\|_r^2 \quad \forall\, n \in \mathbb{N}.$$

Since

$$|\xi_i[\phi]|^2 < (2e_i)^2\|\phi\|_o^2 \leq (2e_i)^2\|\phi\|_r^2 \text{ we get}$$

$$|m^n[\phi]|^2 + Q^n(\phi,\phi) = |m^n[\phi]|^2 + \sum_{i=1}^{N} \int_{\mathbb{R}} a^2 \nu_i^n(da)(\xi_i[\phi])^2$$

$$\leq \text{CONSTANT } \|\phi\|_r^2 \quad \forall\, n \in \mathbb{N};\ \text{i.e. (1) holds with } r_2 = r.$$

Let X_t^n; $n \geq 1$ denote the Φ'-valued processes constructed from m^n and μ^n on p.5/6.

Letting ξ_n denote the solution to

$$d\xi_t^n = -L'\xi_t^n + dX_t^n$$

$$\xi_o^n = 0$$

Theorem I.3 gives the existence of p_T such that

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \eta^{\epsilon,T} \text{ on } D([0,T],\Phi_{-p_T})$$

provided that

$$(24) \qquad \lim_{n\to\infty} \int_{\mathbb{R}} |a|^3 \nu_i^n(da) = 0 \quad \forall\, i = 1,\ldots,N$$

and

$$(25) \qquad \lim_{n\to\infty} \int_{\mathbb{R}} a^2 \nu_i^n(da) = \beta_i^2 \quad \forall\, i = 1,\ldots,N,$$

i.e. the previously considered process η^ϵ can be thought of as the limit of solutions to SDE's with Poisson generated noise.

Physically, this type of weak convergence models a situation in which the individual current stimuli of the neuron arrive very densely in each small time interval so as to create a total contribution to the electrical potential which behaves like the continuous Wiener process.

On the other hand, if (24) and (25) are replaced by

$$\lim_{n\to\infty} \int_{\mathbb{R}} (e^{iay} - 1 - iay)\nu_i^n(da) = \int_{\mathbb{R}} (e^{iay} - 1 - iay)\nu_i^\epsilon(da)$$

for all $y \in \mathbb{R}$, then theorem I.5 gives

$$\xi^{n,T} \underset{n\to\infty}{\Longrightarrow} \xi^{\epsilon,T} \text{ on } D([0,T],\Phi_{-p_T})$$

where ξ^ϵ is the process with mean functional m^ϵ constructed from the Poisson random measure with intensity

$$\mu^{e} = \sum_{i=1}^{N} \nu_{i}^{e} x \delta_{\xi_i}.$$

This latter convergence can be thought of as modelling a situation in which the individual stimuli received by the neuron do not tend to arrive very densely packed in each small time interval, but rather tend to arrive clustered at random points of time.

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